Discussion 15 Worksheet

Vector fields and line integrals

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MATH 53 Multivariable Calculus

1 Plotting Vector Fields

Create a plot of each of the following vector fields.

1. $\vec{F}(x,y) = x\,\vec{i} + y\,\vec{j}$
2. $\vec{F}(x,y) = x\,\vec{i} + x\,\vec{j}$
3. $\vec{F}(x,y) = -\dfrac{y}{x^2+y^2}\,\vec{i} + \dfrac{x}{x^2+y^2}\,\vec{j}$

2 Gradient Vector Fields

For each of the following functions, compute the gradient vector field, and draw this vector field in the plane. How does this vector field relate to the level sets of f?

1. $f(x,y) = x^2 + y^2$
2. $f(x,y) = x^2 - y^2$
3. $f(x,y) = x - y^2$

3 Conservative Vector Fields

For each of the following vector fields $\vec{F}$, either prove that $\vec{F}$ is conservative by finding a function f such that $\nabla f = \vec{F}$, or prove that f is not conservative.

1. $\vec{F}(x,y) = x\,\vec{i} + y\,\vec{j}$
2. $\vec{F}(x,y) = x\,\vec{i} + x\,\vec{j}$
3. $\vec{F}(x,y,z) = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$
4. $\vec{F} = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$

4 Line integrals of functions

Compute the following line integrals:

1. $\int_C x\,ds$ where C is the graph of $f(x) = \frac{1}{2}x^2$ going from $x = 0$ to $x = 2$.
2. $\int_C xy^4\,ds$ where C is the right half of the unit circle.
3. $\int_C x^2 y\,ds$ in 3D where C is given by $x = \cos t, y = \sin t, z = t, 0 \le t \le \pi/2$.

5 Line integrals of vector fields

Compute the following line integrals:

1. $\int_C y^2\,dx + x^2\,dy$ where C is the line segment from $(1,0)$ to $(4,1)$.
2. $\int_C x\,dx + y\,dy + z\,dz$ where C is the straight line connecting $(0,0,0)$ to $(1,2,3)$. Can you figure out what the integral will be when the endpoint of C is an arbitrary point (x_0, y_0, z_0) ?
3. $\int_C \vec{F} \circ d\vec{r}$ where $\vec{F}(x,y) = (y,1)$ and C is the unit circle, traversed counterclockwise. Can you say something about the integral of $\vec{F}_2(x,y) = (y,0)$ along the same curve without doing another computation?

6 Challenge

1. Is the vector field
$$\vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$
conservative (considered over the region R consisting of all points in $\mathbb{R}^2$ other than $(0,0)$)?

Note: These problems are taken from the worksheets for Math 53 in the Spring of 2021 with Prof. Stankova.